INTERNATIONAL JOURNAL OF INFORMATIVE & FUTURISTIC RESEARCH (IJIFR)
ISSN: 2347-1697
VOLUME 13||ISSUE 9|| MAY 2026
An Open Access, Peer Reviewed, Cross Reference DOI & Google Scholar Indexed International Research Journal
UGC Approved || IJIFR Impact Factor= 8.057
Estd Year: 2013 || Email: editors@ijifr.org
Availaible Online at: https://www.ijifr.org/

DOI: https://doi.org/10.64672/IJIFR/26.05.13.09.034 PUBLISHED ON: MAY 24, 2026

A STUDY OF ANALYTICITY OF THE GENERALIZED SUMUDU–FINITE MELLIN TRANSFORM

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ABSTRACT
Integral transforms constitute an essential analytical framework for addressing mathematical models arising in science and engineering, particularly those involving differential and integral equations. In recent years, the Sumudu transform has emerged as an effective alternative to classical transforms due to its operational simplicity and unit-preserving nature, while the Finite Mellin transform has proven useful in problems defined over bounded domains with scale-dependent behaviour. Motivated by the complementary advantages of these two transforms, the present work introduces and examines the Sumudu–Finite Mellin Transform. The transform is formulated in both the classical and distributional settings by constructing suitable testing function spaces through the Gelfand–Shilov methodology. Furthermore, the analytical structure of the proposed transform is investigated, and an analyticity result is established by fixing one of the transform parameters. The developed framework provides a mathematically rigorous foundation for further applications of the Sumudu–Finite Mellin Transform in boundary value problems, partial differential equations, and related areas of applied analysis.

KEYWORDS *Sumudu Transform, Finite Mellin Transform, Distribution Theory, Generalized Function, Testing Function Space, Analyticity*

Recommended Citation:
Rangari, A. N., Gajbe, R. J., Sharma, V.A. : "A Study of analyticity of the Generalized Sumudu–Finite Mellin Transform", International Journal of Informative & Futuristic Research (IJIFR), Vol. (13) (9), May 2026, pp. 1598-1609.
https://doi.org/10.64672/IJIFR/26.05.13.09.034

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1. INTRODUCTION

Integral transforms are indispensable mathematical tools utilized across science and engineering to solve differential, integral, and integro-differential equations. By mapping complex differential operators into simpler algebraic expressions, classical methods like the Fourier and Laplace transforms facilitate a more efficient analysis of physical systems [1], [2]. To address the evolving complexity of modern mathematical models, researchers have introduced specialized transforms designed for greater computational efficiency.

One such development is the Sumudu transform, which is highly valued for its ability to preserve the original units of a function while simplifying the solution process for ordinary and partial differential

equations [3], [4]. In parallel, the Mellin transform and its variant, the Finite Mellin transform, have become essential for analyzing boundary value problems, particularly those defined on finite spatial intervals [5], [6].

Recently, the focus has shifted toward developing hybrid analytical techniques that combine the strengths of different integral operators. These generalized transforms are supported by new inversion theorems, operational formulas, and established properties within distributional spaces [7], [8], [9]. Such advancements have notably broadened the capabilities of transform methods in solving sophisticated models in the applied sciences [10], [11].

The Sumudu–Finite Mellin transform represents a significant step in this evolution. By merging the unit-preserving properties of the Sumudu transform with the spatial efficiency of the Finite Mellin transform, it provides a robust framework for addressing multi-dimensional problems [12], [13]. This combined approach offers an effective methodology for tackling complex equations found in modern physics, engineering, and applied mathematics [14], [15].

2. DEFINITION

The Sumudu integral transform with parameter u of the function $f(t)$ is denoted by

$$S[f(t)] = F(u) = \frac{1}{u}\int_0^\infty e^{-\frac{t}{u}} f(t)dt \quad \text{for parameter } u > 0.$$

The Finite – Mellin integral transform with parameter p of the function $f(x)$ is denoted by

$M_f[f(x)] = F(p)$ and is given by, $\quad M_f[f(x)] = F(p) = \int_0^a \left[\frac{a^{2p}}{x^{p+1}} - x^{p-1}\right] f(x)dx$

whenever this integral exists, for $p > 0$ is the parameter.

The Sumudu – Finite Mellin Transform is defined as,

$$SM_f\{f(t,x)\} = F(u,p) = \int_0^\infty\int_0^a f(t,x)K(t,x)dtdx = \int_0^\infty\int_0^a \frac{1}{u}e^{-\frac{t}{u}}\left[\frac{a^{2p}}{x^{p+1}} - x^{p-1}\right] f(t,x)dtdx$$

where, $K(t,x) = \frac{1}{u}e^{-\frac{t}{u}}\left[\frac{a^{2p}}{x^{p+1}} - x^{p-1}\right]$ is the Kernel of Sumudu – Finite Mellin Transform.

3. VARIOUS TESTING FUNCTION SPACES

3.1. The Space $SM_{f,a,b,c,d,\alpha}$:

Let I be the open set in $R_+ X R_+$ and E_+ denotes the class of infinitely differentiable function defined on I, The Space $SM_{f,a,b,c,d,\alpha}$ is defined as,

$$SM_{f,a,b,c,d,\alpha} = \left\{\phi : \phi \in E_+ / \gamma_{a,b,c,d,q,l}\,\phi(t,x) = \sup_{\substack{0<t<\infty \\ 0<x<a}} \left|K_{a,b}(t)\lambda_{c,d}(x)x^{q+1}D_t^l D_x^q \phi(t,x)\right| \le C_{lq}A^a a^{a\alpha}\right\}$$

for each $l,q = 0,1,2,\ldots$

where, $K_{a,b}(t) = \begin{cases} e^{at} & , \quad 0 \le t < \infty \\ e^{bt} & , \quad -\infty < t < 0 \end{cases}$ and $\lambda_{c,d}(x) = \begin{cases} x^{+c} & , \quad 0 < x < 1 \\ x^{+d} & , \quad 1 < x < a \end{cases}$

are the Kernels for testing function space of Sumudu – Finite Mellin Transform respectively.
Also where, the constants A and C_{lq} depend on the testing function ϕ .

3.2. The Space $SM^{\beta}_{f,a,b,c,d}$:

The Space $SM^{\beta}_{f,a,b,c,d}$ is defined as,

$$SM^{\beta}_{f,a,b,c,d}=\left\{\phi:\phi\in E_{+}/\sigma_{a,b,c,d,q,l}\,\phi(t,x)=\sup_{\substack{0<t<\infty\\0<x<a}}\left|K_{a,b}(t)\lambda_{c,d}(x)x^{q+1}D_t^lD_x^q\phi(t,x)\right|\le C_{aq}B^ll^{l\beta}\right\}$$

where, the constants B and C_{aq} depend on the testing function ϕ.

3.3. The Space $SM^{\beta}_{f,a,b,c,d,\alpha}$:

The Space $SM^{\beta}_{f,a,b,c,d,\alpha}$ is defined as,

$$SM^{\beta}_{f,a,b,c,d,\alpha}=\left\{\phi:\phi\in E_{+}/\rho_{a,b,c,d,q,l}\,\phi(t,x)=\sup_{\substack{0<t<\infty\\0<x<a}}\left|K_{a,b}(t)\lambda_{c,d}(x)x^{q+1}D_t^lD_x^q\phi(t,x)\right|\le CA^{\alpha}a^{a\alpha}B^ll^{l\beta}\right\}$$

where, the constants A,B and C depend on the testing function ϕ.

3.4. The Space $SM_{f,a,b,c,d,\gamma}$:

The Space $SM_{f,a,b,c,d,\gamma}$ is defined as,

$$SM_{f,a,b,c,d,\gamma}=\left\{\phi:\phi\in E_{+}/\xi_{a,b,c,d,q,l}\phi(t,x)=\sup_{\substack{0<t<\infty\\0<x<a}}\left|K_{a,b}(t)\lambda_{c,d}(x)x^{q+1}D_t^lD_x^q\phi(t,x)\right|\le C_{al}A^qa^{q\gamma}\right\}$$

3.5. The Space $SM_{f,a,b,c,d,\alpha,m}$:

The Space $SM_{f,a,b,c,d,\alpha,m}$ is defined as,

$$SM_{f,a,b,c,d,\alpha,m}=\left\{\phi:\phi\in E_{+}/\gamma_{a,b,c,d,q,l}\,\phi(t,x)=\sup_{\substack{0<t<\infty\\0<x<a}}\left|K_{a,b}(t)\lambda_{c,d}(x)x^{q+1}D_t^lD_x^q\phi(t,x)\right|\le C_{lq\delta}\left(m+\delta\right)^a\,a^{a\alpha}\right\}$$

for any $\delta>0$, where 'm' is the constant depending on the testing function ϕ.

3.6. The Space $SM^{\beta,n}_{f,a,b,c,d}$:

The Space $SM^{\beta,n}_{f,a,b,c,d}$ is defined as,

$$SM^{\beta,n}_{f,a,b,c,d}=\left\{\phi:\phi\in E_{+}/\sigma_{a,b,c,d,q,l}\phi(t,x)=\sup_{\substack{0<t<\infty\\0<x<a}}\left|K_{a,b}(t)\lambda_{c,d}(x)x^{q+1}D_t^lD_x^q\phi(t,x)\right|\le C_{aq\varepsilon}\left(n+\varepsilon\right)^l\,l^{l\beta}\right\}$$

for any $\varepsilon > 0$, where 'n' is the constant depending on the testing function ϕ.

3.7. The Space $SM^{\beta,n}_{f,a,b,c,d,\alpha,m}$:

The Space $SM^{\beta,n}_{f,a,b,c,d,\alpha,m}$ is defined as,

$$SM^{\beta,n}_{f,a,b,c,d,\alpha,m} = \left\{ \phi : \phi \in E_+ \,/\, \rho_{a,b,c,d,q,l}\phi(t,x) = \sup_{\substack{0<t<\infty \\ 0<x<a}} \left| K_{a,b}(t)\lambda_{c,d}(x)x^{q+1}D_t^l D_x^q \phi(t,x) \right| \le C_{q\varepsilon}(m+\delta)^a (n+\varepsilon)^l a^{a\alpha} l^{l\beta} \right\}$$

for any $\delta > 0$, $\varepsilon > 0$ and for given $m > 0$ and $n > 0$.

3.8. The Space $SM_{f,a,b,c,d,\gamma,p}$:

The Space $SM_{f,a,b,c,d,\gamma,p}$ is defined as,

$$SM_{f,a,b,c,d,\gamma,p} = \left\{ \phi : \phi \in E_+ \,/\, \xi_{a,b,c,d,q,l}\phi(t,x) = \sup_{\substack{0<t<\infty \\ 0<x<a}} \left| K_{a,b}(t)\lambda_{c,d}(x)x^{q+1}D_t^l D_x^q \phi(t,x) \right| \le C_{al\gamma}(p+\gamma)^q q^{q\gamma} \right\}$$

for any $\gamma > 0$, where 'p' is the constant depending on the testing function ϕ.

3.9. The Space $S^{\nu}M_{f,a,b,c,d,\alpha}$:

The Space $S^{\nu}M_{f,a,b,c,d,\alpha}$ is defined as,

$$S^{\nu}M_{f,a,b,c,d,\alpha} = \left\{ \phi : \phi \in E_- \,/\, i_{a,b,c,d,q,l}\phi(t,x) = \sup_{\substack{-\infty<t<0 \\ 0<x<a}} \left| K_{a,b}(-t)\lambda_{c,d}(x)x^{q+1}D_t^l D_x^q \phi(t,x) \right| \le C_{lq}A^a a^{a\alpha} \right\}$$

The smooth function $\phi(t,x)$ defined on I_2 is in $S^{\nu}M_{f,a,b,c,d,\alpha}$ if $\phi^{\nu}(t,x) = \phi(-t,x)$ is in $SM_{f,a,b,c,d,\alpha}$.

3.10. The Space $S^{\nu}M^{\beta}_{f,a,b,c,d}$:

The Space $S^{\nu}M^{\beta}_{f,a,b,c,d}$ is defined as,

$$S^{\nu}M^{\beta}_{f,a,b,c,d} = \left\{ \phi : \phi \in E_- \,/\, j_{a,b,c,d,q,l}\phi(t,x) = \sup_{\substack{-\infty<t<0 \\ 0<x<a}} \left| K_{a,b}(-t)\lambda_{c,d}(x)x^{q+1}D_t^l D_x^q \phi(t,x) \right| \le C_{aq}B^l l^{l\beta} \right\}$$

Here, $\phi^{\nu}(t,x) = \phi(-t,x)$ is in $SM^{\beta}_{f,a,b,c,d}$.

3.11. The Space $S^{\nu}M^{\beta}_{f,a,b,c,d,\alpha}$:

The Space $S^{\nu}M^{\beta}_{f,a,b,c,d,\alpha}$ is defined as,

$$S^{\nu}M^{\beta}_{f,a,b,c,d,\alpha} = \left\{ \phi : \phi \in E_- \,/\, \mu_{a,b,c,d,q,l}\phi(t,x) = \sup_{\substack{-\infty<t<0 \\ 0<x<a}} \left| K_{a,b}(-t)\lambda_{c,d}(x)x^{q+1}D_t^l D_x^q \phi(t,x) \right| \le CA^{\alpha} a^{a\alpha} B^l l^{l\beta} \right\}$$

where, the constants A, B and C depend on the testing function ϕ.

3.12. The Space $S^{\nu}M^{\beta}_{f,a,b,c,d,\gamma}$:

The Space $S^{\nu}M^{\beta}_{f,a,b,c,d,\alpha,\gamma}$ is defined as,

$$S^{\nu}M^{\beta}_{f,a,b,c,d,\alpha,\gamma}=\left\{\phi:\phi\in E_{-}/\theta_{a,b,c,d,q,l}\,\phi(t,x)=\sup_{\substack{0<t<\infty\\-a<x<0}}\left|K_{a,b}(t)\lambda_{c,d}(-x)(-x)^{q+1}D_t^lD_x^q\phi(t,x)\right|\le C_{al}A^qq^{q\gamma}\right\}$$

4. DISTRIBUTIONAL GENERALIZED SUMUDU – FINITE MELLIN TRANSFORM $\left(SM_fT\right)$:

For $f(t,x)\in SM^{*\beta}_{f,a,b,c,d,\alpha,}$ where $SM^{*\beta}_{f,a,b,c,d,\alpha}$ is the dual space of $SM^{\beta}_{f,a,b,c,d,\alpha.}$ It contains all distributions of compact support. The distributional Sumudu – Finite Mellin Transform is a function of $f(t,x)$ is defined as,

$$SM_f\left\{f(t,x)\right\}=F(u,p)=\left\langle f(t,x),\frac{1}{u}e^{-\frac{t}{u}}\left[\frac{a^{2p}}{x^{p+1}}-x^{p-1}\right]\right\rangle$$

$-(4.1)$

Where, for each fixed $t\,(0<t<\infty),x\,(0<x<a),u>0$ and $p>0,$ the right-hand side of (4.1) has a sense as an application of $f(t,x)\in SM^{*\beta}_{f,a,b,c,d,\alpha}$ to $\frac{1}{u}e^{-\frac{t}{u}}\left[\frac{a^{2p}}{x^{p+1}}-x^{p-1}\right]\in SM^{\beta}_{f,a,b,c,d,\alpha.}$

5. ANALYTICITY THEOREM

Let $f(t,x)\in SM^{*\beta}_{f,a,b,c,d,\alpha}$ and its Sumudu–Finite Mellin transform $F(u,p)$ is defined by

$$SM_f\left\{f(t,x)\right\}=F(u,p)=\left\langle f(t,x),\frac{1}{u}e^{-\frac{t}{u}}\left[\frac{a^{2p}}{x^{p+1}}-x^{p-1}\right]\right\rangle$$

then $F(u,p)$ is analytic for some fixed $u>0,p>0$ on Ω_f where $\Omega_f=\left\{(u,p):\sigma_1<u<\sigma_2\right\}$ and

$$i)D_uF(u,p)=\left\langle f(t,x),\left(-\frac{1}{u^2}e^{-\frac{t}{u}}+\frac{t}{u^3}e^{-\frac{t}{u}}\right)\left(\frac{a^{2p}}{x^{p+1}}-x^{p-1}\right)\right\rangle$$

$-$ (5.1)

$$ii)D_pF(u,p)=\left\langle f(t,x),\frac{1}{u}e^{-\frac{t}{u}}\left[\frac{a^{2p}}{x^{p+1}}\left(2\ln a-\ln x\right)-x^{p-1}\left(\ln x\right)\right]\right\rangle$$

$-$ (5.2)

Proof: Let u be an arbitrary but fixed point in Ω_f. Choose the positive real numbers a,b and r such that $\sigma_1<a<u-r<u+r<b<\sigma_2.$

Also let Δu be a complex increment such that $0<\left|\Delta u\right|<r.$

Consider, for $\Delta u\neq 0,$ we have

$$\frac{F(u+\Delta u,p)-F(u,p)}{\Delta u}-\left\langle f(t,x),\frac{\partial}{\partial u}\frac{e^{-\frac{t}{u}}}{u}\left(\frac{a^{2p}}{x^{p+1}}-x^{p-1}\right)\right\rangle$$

$$=\frac{1}{\Delta u}\left\{\left\langle f(t,x),\frac{e^{-\frac{t}{u+\Delta u}}}{u+\Delta u}\left(\frac{a^{2p}}{x^{p+1}}-x^{p-1}\right)\right\rangle-\left\langle f(t,x),\frac{e^{-\frac{t}{u}}}{u}\left(\frac{a^{2p}}{x^{p+1}}-x^{p-1}\right)\right\rangle\right\}-\left\langle f(t,x),\frac{\partial}{\partial u}\frac{e^{-\frac{t}{u}}}{u}\left(\frac{a^{2p}}{x^{p+1}}-x^{p-1}\right)\right\rangle$$

$$=\frac{1}{\Delta u}\left\langle f(t,x),\left\{\frac{e^{-\frac{t}{u+\Delta u}}}{u+\Delta u}-\frac{e^{-\frac{t}{u}}}{u}\right\}\left(\frac{a^{2p}}{x^{p+1}}-x^{p-1}\right)\right\rangle-\left\langle f(t,x),\frac{\partial}{\partial u}\frac{e^{-\frac{t}{u}}}{u}\left(\frac{a^{2p}}{x^{p+1}}-x^{p-1}\right)\right\rangle$$

$$=\left\langle f(t,x),\frac{1}{\Delta u}\left\{\frac{e^{-\frac{t}{u+\Delta u}}}{u+\Delta u}-\frac{e^{-\frac{t}{u}}}{u}\right\}\left(\frac{a^{2p}}{x^{p+1}}-x^{p-1}\right)\right\rangle-\left\langle f(t,x),\frac{\partial}{\partial u}\frac{e^{-\frac{t}{u}}}{u}\left(\frac{a^{2p}}{x^{p+1}}-x^{p-1}\right)\right\rangle$$

$$=\left\langle f(t,x),\frac{1}{\Delta u}\left\{\frac{e^{-\frac{t}{u+\Delta u}}}{u+\Delta u}-\frac{e^{-\frac{t}{u}}}{u}\right\}\left(\frac{a^{2p}}{x^{p+1}}-x^{p-1}\right)-\frac{\partial}{\partial u}\frac{e^{-\frac{t}{u}}}{u}\left(\frac{a^{2p}}{x^{p+1}}-x^{p-1}\right)\right\rangle$$

$$\frac{F(u+\Delta u,p)-F(u,p)}{\Delta u}-\left\langle f(t,x),\frac{\partial}{\partial u}\frac{e^{-\frac{t}{u}}}{u}\left(\frac{a^{2p}}{x^{p+1}}-x^{p-1}\right)\right\rangle=\left\langle f(t,x),\psi_{\Delta u}(t,x)\right\rangle$$

where, $\psi_{\Delta u}(t,x)=\frac{1}{\Delta u}\left\{\frac{e^{-\frac{t}{u+\Delta u}}}{u+\Delta u}-\frac{e^{-\frac{t}{u}}}{u}\right\}\left(\frac{a^{2p}}{x^{p+1}}-x^{p-1}\right)-\frac{\partial}{\partial u}\frac{e^{-\frac{t}{u}}}{u}\left(\frac{a^{2p}}{x^{p+1}}-x^{p-1}\right)$

$-(5.3)$

To prove that, $\psi_{\Delta u}(t,x)\in SM^{\beta}_{f,a,b,c,d,\alpha}$, we shall show that $|\Delta u|\to 0,\psi_{\Delta u}(t,x)$ converges in $SM^{\beta}_{f,a,b,c,d,\alpha}$ to zero.

To proceed, let C denotes the circle with centre at u and radius r_1.

where $0<r<r_1<\min(u-a,b-u)$.

We may interchange differentiation on u with differentiation on t.

By using Cauchy integral formula, we get

$$(-D_t)^l\psi_{\Delta u}(t,x)=\left(\frac{a^{2p}}{x^{p+1}}-x^{p-1}\right)(-1)^l\left[\frac{1}{\Delta u}\frac{e^{-\frac{t}{u+\Delta u}}}{(u+\Delta u)^{l+1}}-\frac{e^{-\frac{t}{u}}}{u^{l+1}}-\frac{\partial}{\partial u}\left(\frac{e^{-\frac{t}{u}}}{u^{l+1}}\right)\right]$$

Now applying Cauchy's Integral formula, we get

INTERNATIONAL JOURNAL OF INFORMATIVE & FUTURISTIC RESEARCH (IJIFR)
ISSN: 2347-1697
IJIFR /20260524/V13/E9/034
VOLUME 13||ISSUE 9|| MAY 2026
https://www.ijifr.org/

$$(-D_t)^l \psi_{\Delta u}(t,x) = \frac{1}{\Delta u}\left(\frac{a^{2p}}{x^{p+1}} - x^{p-1}\right)\left[\frac{1}{2\pi i}\int_C \frac{z^{-(l+1)}e^{-\frac{t}{z}}}{z-u-\Delta u}dz - \frac{1}{2\pi i}\int_C \frac{z^{-(l+1)}e^{-\frac{t}{z}}}{z-u}dz\right]$$

$$-\left(\frac{a^{2p}}{x^{p+1}} - x^{p-1}\right)\frac{1}{2\pi i}\int_C \frac{z^{-(l+1)}e^{-\frac{t}{z}}}{(z-u)^2}dz$$

$$= \frac{1}{\Delta u}\left(\frac{a^{2p}}{x^{p+1}} - x^{p-1}\right)\left[\frac{1}{2\pi i}\int_C \left(\frac{1}{z-u-\Delta u} - \frac{1}{z-u}\right)z^{-(l+1)}e^{-\frac{t}{z}}dz\right] - \left(\frac{a^{2p}}{x^{p+1}} - x^{p-1}\right)\frac{1}{2\pi i}\int_C \frac{z^{-(l+1)}e^{-\frac{t}{z}}}{(z-u)^2}dz$$

$$= \frac{1}{\Delta u}\left(\frac{a^{2p}}{x^{p+1}} - x^{p-1}\right)\left[\frac{1}{2\pi i}\int_C \left(\frac{(z-u)-(z-u-\Delta u)}{(z-u-\Delta u)(z-u)}\right)z^{-(l+1)}e^{-\frac{t}{z}}dz\right] - \left(\frac{a^{2p}}{x^{p+1}} - x^{p-1}\right)\frac{1}{2\pi i}\int_C \frac{z^{-(l+1)}e^{-\frac{t}{z}}}{(z-u)^2}dz$$

$$= \frac{1}{\Delta u}\left(\frac{a^{2p}}{x^{p+1}} - x^{p-1}\right)\left[\frac{1}{2\pi i}\int_C \left(\frac{\Delta u}{(z-u-\Delta u)(z-u)}\right)z^{-(l+1)}e^{-\frac{t}{z}}dz\right] - \left(\frac{a^{2p}}{x^{p+1}} - x^{p-1}\right)\frac{1}{2\pi i}\int_C \frac{z^{-(l+1)}e^{-\frac{t}{z}}}{(z-u)^2}dz$$

$$= \left(\frac{a^{2p}}{x^{p+1}} - x^{p-1}\right)\frac{1}{2\pi i}\int_C \left(\frac{z^{-(l+1)}e^{-\frac{t}{z}}dz}{(z-u-\Delta u)(z-u)}\right) - \left(\frac{a^{2p}}{x^{p+1}} - x^{p-1}\right)\frac{1}{2\pi i}\int_C \frac{z^{-(l+1)}e^{-\frac{t}{z}}}{(z-u)^2}dz$$

$$= \left(\frac{a^{2p}}{x^{p+1}} - x^{p-1}\right)\frac{1}{2\pi i}\int_C \left(\frac{1}{(z-u-\Delta u)(z-u)} - \frac{1}{(z-u)^2}\right)z^{-(l+1)}e^{-\frac{t}{z}}dz$$

$$= \left(\frac{a^{2p}}{x^{p+1}} - x^{p-1}\right)\frac{1}{2\pi i}\int_C \frac{(z-u)-(z-u-\Delta u)}{(z-u-\Delta u)(z-u)^2}z^{-(l+1)}e^{-\frac{t}{z}}dz$$

$$= \left(\frac{a^{2p}}{x^{p+1}} - x^{p-1}\right)\frac{1}{2\pi i}\int_C \frac{\Delta u}{(z-u-\Delta u)(z-u)^2}z^{-(l+1)}e^{-\frac{t}{z}}dz$$

$$(-D_t)^l \psi_{\Delta u}(t,x) = \left(\frac{a^{2p}}{x^{p+1}} - x^{p-1}\right)\frac{\Delta u}{2\pi i}\int_C \frac{z^{-(l+1)}e^{-\frac{t}{z}}}{(z-u-\Delta u)(z-u)^2}dz$$

Now,

$$(D_x)^q (-D_t)^l \psi_{\Delta u}(t,x) = \left[a^{2p}P(-p-q)x^{-p-q-1} - P(p-q)x^{p-q-1}\right]\frac{\Delta u}{2\pi i}\int_C \frac{z^{-(l+1)}e^{-\frac{t}{z}}}{(z-u-\Delta u)(z-u)^2}dz$$

Where, $P(-p-q)$ is a polynomial in $(-p-q)$ etc.

Now, for all $z \in C$ and $0 < t < \infty$,

$$\underset{I}{Sup} \left| K_{a,b}(t)\lambda_{c,d}(x)x^{q+1}\left[a^{2p}P(-p-q)x^{-p-q-1} - P(p-q)x^{p-q-1}\right]\right| \le K$$

Where K is a constant independent of z and t.

Moreover, $|z-u-\Delta u| > r_1 - r > 0$ and $|z-u| = r_1$

Let $C_1 = \max\left\{|z|^{-(l+1)} e^{-\frac{t}{z}}, z \in C\right\}$

Consequently,

$$\underset{I}{Sup} \left| K_{a,b}(t)\lambda_{c,d}(x)x^{q+1} D_t^l D_x^q \psi_{\Delta u}(t,x)\right|$$

$$= \underset{I}{Sup} \left| K_{a,b}(t)\lambda_{c,d}(x)x^{q+1}\left\{\frac{\Delta u}{2\pi i}\int_C \frac{\left[a^{2p}P(-p-q)x^{-p-q-1} - P(p-q)x^{p-q-1}\right]}{(z-u-\Delta u)(z-u)^2} z^{-(l+1)} e^{-\frac{t}{z}} dz\right\}\right|$$

$$\le \frac{|\Delta u|}{2\pi}\int_C \frac{KC_1}{(r_1 - r)(r_1)^2}|dz|$$

$$\le \frac{|\Delta u|}{2\pi}\int_C \frac{C_2}{(r_1 - r)(r_1)^2} 2\pi r_1 \qquad \text{where, } C_2 = KC_1$$

$$\le |\Delta u| \frac{C_2}{(r_1 - r)r_1}$$

The right-hand side is independent of t and converges to zero as $|\Delta u| \to 0$.

This shows that $\psi_{\Delta u}(t,x)$ converges to zero as $|\Delta u| \to 0$.

Let p be an arbitrary but fixed point. Choose the real positive numbers a_2, b_2 and h such that $\sigma_1 < a_2 < \mathrm{Re}\, p - h < \mathrm{Re}\, p + h < b_2 < \sigma_2$.

Also let Δp be a complex increment such that $0 < |\Delta p| < h$.

Consider, for $\Delta p \ne 0$, we write

$$\frac{F(u,p+\Delta p) - F(u,p)}{\Delta p} - \left\langle f(t,x), \frac{\partial}{\partial p}\frac{e^{-\frac{t}{u}}}{u}\left[\frac{a^{2p}}{x^{p+1}} - x^{p-1}\right]\right\rangle = \frac{1}{\Delta p}\left\{\left\langle f(t,x), \frac{e^{-\frac{t}{u}}}{u}\left[\frac{a^{2(p+\Delta p)}}{x^{(p+\Delta p)+1}} - x^{(p+\Delta p)-1}\right]\right\rangle - \left\langle\right.\right.$$

$$= \frac{1}{\Delta p}\left\langle f(t,x), \frac{e^{-\frac{t}{u}}}{u}\left\{\left[\frac{a^{2(p+\Delta p)}}{x^{(p+\Delta p)+1}} - x^{(p+\Delta p)-1}\right] - \left[\frac{a^{2p}}{x^{p+1}} - x^{p-1}\right]\right\}\right\rangle - \left\langle f(t,x), \frac{\partial}{\partial p}\frac{e^{-\frac{t}{u}}}{u}\left[\frac{a^{2p}}{x^{p+1}} - x^{p-1}\right]\right\rangle$$

$$= \left\langle f(t,x), \frac{1}{\Delta p}\frac{e^{-\frac{t}{u}}}{u}\left\{\left[\frac{a^{2(p+\Delta p)}}{x^{(p+\Delta p)+1}} - x^{(p+\Delta p)-1}\right] - \left[\frac{a^{2p}}{x^{p+1}} - x^{p-1}\right]\right\}\right\rangle - \left\langle f(t,x), \frac{\partial}{\partial p}\frac{e^{-\frac{t}{u}}}{u}\left[\frac{a^{2p}}{x^{p+1}} - x^{p-1}\right]\right\rangle$$

$$= \left\langle f(t,x), \frac{1}{\Delta p}\frac{e^{-\frac{t}{u}}}{u}\left\{\left[\frac{a^{2(p+\Delta p)}}{x^{(p+\Delta p)+1}} - x^{(p+\Delta p)-1}\right] - \left[\frac{a^{2p}}{x^{p+1}} - x^{p-1}\right]\right\} - \frac{\partial}{\partial p}\frac{e^{-\frac{t}{u}}}{u}\left[\frac{a^{2p}}{x^{p+1}} - x^{p-1}\right]\right\rangle$$

$$= \left\langle f(t,x), \psi_{\Delta p}(t,x)\right\rangle$$

where, $\psi_{\Delta p}(t,x) = \frac{1}{\Delta p}\frac{e^{-\frac{t}{u}}}{u}\left\{\left[\frac{a^{2(p+\Delta p)}}{x^{(p+\Delta p)+1}} - x^{(p+\Delta p)-1}\right] - \left[\frac{a^{2p}}{x^{p+1}} - x^{p-1}\right]\right\} - \frac{\partial}{\partial p}\frac{e^{-\frac{t}{u}}}{u}\left[\frac{a^{2p}}{x^{p+1}} - x^{p-1}\right]$

$-(5.4)$

To prove that $\psi_{\Delta p}(t,x) \in SM^{\beta}_{f,a,b,c,d,\alpha}$. We shall show that as $|\Delta p| \to 0, \psi_{\Delta p}(t,x)$ converges to zero in $SM^{\beta}_{f,a,b,c,d,\alpha}$.

To proceed, let C denotes the circle with centre at p and radius h_1.

where, $0 < h < h_1 < \min(p - a_2, b_2 - p)$.

We may interchange differentiation on p with differentiation on x and by using Cauchy integral formula,

$$(D_x)^q \psi_{\Delta p}(t,x) = \frac{1}{\Delta p}\frac{e^{-\frac{t}{u}}}{u}\left\{\left[a^{2(p+\Delta p)} P\left(-(p+\Delta p)-q\right)x^{-(p+\Delta p)-q-1} - P\left((p+\Delta p)-q\right)x^{(p+\Delta p)-q-1}\right]\right\}$$

$$-\frac{1}{\Delta p}\frac{e^{-\frac{t}{u}}}{u}\left[a^{2p}P(-p-q)x^{-p-q-1} - P(p-q)x^{p-q-1}\right] - \frac{\partial}{\partial p}\frac{e^{-\frac{t}{u}}}{u}\left[a^{2p}P(-p-q)x^{-p-q-1} - P(p-q)x^{p-q-1}\right]$$

where, $P(-p-q)$ is a polynomial in $(-p-q)$ etc.

Now, applying Cauchy's integral formula, we get

$$(D_x)^q \psi_{\Delta p}(t,x) = \frac{1}{\Delta p}\frac{e^{-\frac{t}{u}}}{u}\frac{1}{2\pi i}\int_C \left[\frac{\left[a^{2z}P(-z-q)x^{-z-q-1} - P(z-q)x^{z-q-1}\right]}{z-p-\Delta p}\right]dz$$

$$-\frac{1}{\Delta p}\frac{e^{-\frac{t}{u}}}{u}\frac{1}{2\pi i}\int_C \left[\frac{a^{2z}P(-z-q)x^{-z-q-1} - P(z-q)x^{z-q-1}}{z-p}\right]dz$$

$$-\frac{e^{-\frac{t}{u}}}{u}\frac{1}{2\pi i}\int_C \left[\frac{a^{2z}P(-z-q)x^{-z-q-1} - P(z-q)x^{z-q-1}}{(z-p)^2}\right]dz$$

INTERNATIONAL JOURNAL OF INFORMATIVE & FUTURISTIC RESEARCH (IJIFR)
ISSN: 2347-1697
IJIFR /20260524/V13/E9/034
VOLUME 13||ISSUE 9|| MAY 2026
https://www.ijifr.org/

$$(D_x)^q \psi_{\Delta p}(t,x) = \frac{1}{\Delta p}\frac{e^{-\frac{t}{u}}}{u}\frac{1}{2\pi i}\int_C \left\{\left[\frac{1}{z-p-\Delta p}-\frac{1}{z-p}\right]\left[a^{2z}P(-z-q)x^{-z-q-1}-P(z-q)x^{z-q-1}\right]\right\}dz$$

$$-\frac{e^{-\frac{t}{u}}}{u}\frac{1}{2\pi i}\int_C \left[\frac{a^{2z}P(-z-q)x^{-z-q-1}-P(z-q)x^{z-q-1}}{(z-p)^2}\right]dz$$

$$(D_x)^q \psi_{\Delta p}(t,x) = \frac{1}{\Delta p}\frac{e^{-\frac{t}{u}}}{u}\frac{1}{2\pi i}\int_C \left\{\left[\frac{z-p-z+p+\Delta p}{(z-p-\Delta p)(z-p)}\right]\left[a^{2z}P(-z-q)x^{-z-q-1}-P(z-q)x^{z-q-1}\right]\right\}dz$$

$$-\frac{e^{-\frac{t}{u}}}{u}\frac{1}{2\pi i}\int_C \left[\frac{a^{2z}P(-z-q)x^{-z-q-1}-P(z-q)x^{z-q-1}}{(z-p)^2}\right]dz$$

$$(D_x)^q \psi_{\Delta p}(t,x) = \frac{1}{\Delta p}\frac{e^{-\frac{t}{u}}}{u}\frac{1}{2\pi i}\int_C \left\{\left[\frac{\Delta p}{(z-p-\Delta p)(z-p)}\right]\left[a^{2z}P(-z-q)x^{-z-q-1}-P(z-q)x^{z-q-1}\right]\right\}dz$$

$$-\frac{e^{-\frac{t}{u}}}{u}\frac{1}{2\pi i}\int_C \left[\frac{a^{2z}P(-z-q)x^{-z-q-1}-P(z-q)x^{z-q-1}}{(z-p)^2}\right]dz$$

$$(D_x)^q \psi_{\Delta p}(t,x) = \frac{e^{-\frac{t}{u}}}{u}\frac{1}{2\pi i}\int_C \left\{\left[\frac{1}{(z-p-\Delta p)(z-p)}\right]\left[a^{2z}P(-z-q)x^{-z-q-1}-P(z-q)x^{z-q-1}\right]\right\}dz$$

$$-\frac{e^{-\frac{t}{u}}}{u}\frac{1}{2\pi i}\int_C \left[\frac{a^{2z}P(-z-q)x^{-z-q-1}-P(z-q)x^{z-q-1}}{(z-p)^2}\right]dz$$

$$=\frac{e^{-\frac{t}{u}}}{u}\frac{1}{2\pi i}\int_C \left\{\left[\frac{1}{(z-p-\Delta p)(z-p)}-\frac{1}{(z-p)^2}\right]\left[a^{2z}P(-z-q)x^{-z-q-1}-P(z-q)x^{z-q-1}\right]\right\}dz$$

$$(D_x)^q \psi_{\Delta p}(t,x) = \frac{e^{-\frac{t}{u}}}{u}\frac{1}{2\pi i}\int_C \left\{\left[\frac{z-p-z+p+\Delta p}{(z-p-\Delta p)(z-p)^2}\right]\left[a^{2z}P(-z-q)x^{-z-q-1}-P(z-q)x^{z-q-1}\right]\right\}dz$$

$$(D_x)^q \psi_{\Delta p}(t,x) = \frac{e^{-\frac{t}{u}}}{u}\frac{\Delta p}{2\pi i}\int_C \left\{\frac{a^{2z}P(-z-q)x^{-z-q-1}-P(z-q)x^{z-q-1}}{(z-p-\Delta p)(z-p)^2}\right\}dz$$

$$\left(-D_t\right)^l \left(D_x\right)^q \psi_{\Delta p}(t,x) = (-1)^l u^{-(l+1)} e^{-\frac{t}{u}} \frac{\Delta p}{2\pi i} \int_C \left\{ \frac{a^{2z} P(-z-q) x^{-z-q-1} - P(z-q) x^{z-q-1}}{\left(z-p-\Delta p\right)\left(z-p\right)^2} \right\} dz$$

Now, for all $z \in C$ and $0 < x < a$.

$$\sup_I \left| K_{a,b}(t) \lambda_{c,d}(x) x^{q+1} u^{-(l+1)} e^{-\frac{t}{u}} \right| \le K$$

where, K is a constant independent of z and x.

Moreover, $\left|\left(z-p-\Delta p\right)\right| > h_1 - h > 0$ and $\left|z-p\right| = h_1$

$$C_1 = max\left\{ a^{2z} P\left(-z-q\right) x^{-z-q-1} - P(z-q) x^{z-q-1}, z \in C \right\}$$

Consequently,

$$\underset{I}{Sup} \left| K_{a,b}(t) \lambda_{c,d}(x) x^{q+1} \left(D_t\right)^l \left(D_x\right)^q \psi_{\Delta p}(t,x) \right|$$

$$= \underset{I}{Sup} \left| K_{a,b}(t) \lambda_{c,d}(x) x^{q+1} \frac{e^{-\frac{t}{u}}}{u^{(l+1)}} \frac{\Delta p}{2\pi i} \int_C \left\{ \frac{a^{2z} P(-z-q) x^{-z-q-1} - P(z-q) x^{z-q-1}}{\left(z-p-\Delta p\right)\left(z-p\right)^2} \right\} dz \right|$$

$$\le \frac{|\Delta p|}{2\pi} \int_C \left[\frac{KC_1}{\left(h_1 - h\right)\left(h_1\right)^2} \right] |dz|$$

$$\le \frac{|\Delta p|}{2\pi} \int_C \left[\frac{C_2}{\left(h_1 - h\right)\left(h_1\right)^2} \right] 2\pi h_1, \qquad \text{where } C_2 = KC_1$$

$$\le \frac{|\Delta p| C_2}{\left(h_1 - h\right) h_1}$$

The right – hand side is independent of and converges to zero as $|\Delta p| \to 0$. This shows that $\psi_{\Delta p}(t,x)$ converges to zero as $|\Delta p| \to 0$, which ends the proof.

6. CONCLUSION

In this study, the analytical properties of the Sumudu transform and the finite Mellin transform have been investigated. The results demonstrate that these transforms possess well-defined operational and analytical structures and can be effectively applied in suitable function spaces. The Sumudu transform provides a useful alternative to the Laplace transform due to its unit-preserving property, while the finite Mellin transform is particularly useful for functions defined on finite intervals. The established results indicate that these transforms are valuable tools for solving differential equations and analyzing problems in applied mathematics, mathematical physics, and engineering.

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